

# Kronecker products of projective representations of translation groups\*

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## Abstract

Irreducible projective representations of the translation group of a finite  $N \times N$  two-dimensional lattice can be labeled by symbols  $\langle n, l; \mathbf{q} \rangle$ , where  $N = \nu n$ ,  $\gcd(l, n) = 1$  and  $\mathbf{q}$  denotes an irreducible representation of  $Z_\nu^2$ . Obtained matrices are  $n$ -dimensional and the factor system of this representation does not depend on  $\mathbf{q}$  and equals  $m_n^{(l)}([n_1, n_2], [n'_1, n'_2]) = \exp(2\pi i l n_2 n'_1 / N)$ . For a given  $n$  the  $N \times N$  lattice can be viewed as a  $\nu \times \nu$  lattice consisting of  $n \times n$  magnetic cells. The Kronecker product of such representations is another projective representation which can be decomposed into irreducible ones. It is interesting that such product can lead to the magnetic periodicity different from  $n$ ,  $n'$  and even  $nn'$  or  $\text{lcm}(n, n')$ . For example, a product  $\langle n, l; \mathbf{q} \rangle \otimes \langle n, l; \mathbf{q}' \rangle$  for *even*  $N$  decomposes into representations  $\langle \frac{n}{2}, l; \mathbf{k} \rangle$ : there are four representations with  $k_i - q_i - q'_i = 0, \nu$  and each of them appears  $n/2$  times. Similarly, the coupling of  $d$  representations  $\langle n, 1; \mathbf{q}^{(j)} \rangle$ ,  $j = 1, 2, \dots, d$  with  $n = dM$  changes the magnetic period from  $n$  to  $M$ . It is caused by multiplication of the *charge* by  $d$  what corresponds to a system of  $d$  electrons.

## 1 Introduction

Projective (or ray) representations investigated by Schur at the beginning of this century [1] are widely applied in quantum mechanics (due to factor systems related with them) and crystallography (especially as a tool in construction of space group representations). In the sixties Brown [2] applied them to investigation of movement of a Bloch electron in a magnetic field. Almost at the same time Zak [3] proposed an equivalent approach: ray representations of the translation group were considered as vector (*i.e.* ordinary) representations of its covering group being in fact a central extension of the translation group and a group of factors (see also [4]). The problem investigated by these authors is strongly related to the Landau quantization and, therefore, to the quantum Hall effect (see, for example, articles by Zak [5], Dana, Avron and Zak [6], Aoki [7]). Many authors, however, rejected some representations, obtained in the mathematical analysis of this problem, and claimed that they are ‘nonphysical’ [3].

In the series of articles [4, 8] the magnetic translation group was studied within the frame proposed by Zak, *i.e.* as a central extension of the translation group.

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These investigations gave a basis to construct and consider all representations, including those called ‘nonphysical’ [9]. Moreover, the physical relevance and possible applications of such representations were indicated.

In this article the magnetic translation operators are studied applying Brown’s approach, *i.e.* projective representations of the (two-dimensional) translation group are considered. In Sec. 2, after recalling Brown’s definitions, matrix elements of finite irreducible projective representations are given (they have been obtained from those introduced by Brown applying a gauge transformation). In the next section tensor (Kronecker) products of such representations are investigated.

## 2 Irreducible projective representations

Brown [2] introduced magnetic translation operators as

$$T(\mathbf{R}) = \exp[-i\mathbf{R} \cdot (\mathbf{p} - e\mathbf{A}/c)/\hbar]. \quad (1)$$

These operators commute with the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}/c)^2 + V(\mathbf{r}), \quad (2)$$

describing an electron in a periodic potential  $V(\mathbf{r})$  and a uniform magnetic field

$$\mathbf{H} = \text{rot } \mathbf{A}, \quad \text{where} \quad \mathbf{A} = \frac{1}{2}(\mathbf{H} \times \mathbf{r}). \quad (3)$$

The introduced operators form a projective representation of the crystal translation groups, what is expressed by the following relation:

$$T(\mathbf{R})T(\mathbf{R}') = T(\mathbf{R} + \mathbf{R}')m'(\mathbf{R}, \mathbf{R}'), \quad (4)$$

where

$$m'(\mathbf{R}, \mathbf{R}') = \exp[-i(\mathbf{R} \times \mathbf{R}') \cdot \mathbf{h}/2] \quad (5)$$

is a factor system of this representation with  $\mathbf{h} = e\mathbf{H}/\hbar c$ .

Imposing the periodic conditions Brown showed (see also [3]) that the magnetic field can be assumed to equal

$$\mathbf{h} = \frac{2\pi}{N} \frac{L}{\Omega} \mathbf{a}_3 \quad (6)$$

for an integer  $L$  mutually prime with  $N$ ;  $\Omega = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3$  is the volume of the primitive cell,  $\mathbf{a}_i$  are the primitive translations and  $N$  the period of the crystal lattice. For such a choice of  $\mathbf{h}$  [and factor system (5)] Brown obtained  $N$ -dimensional irreducible projective representations for  $\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$  with matrix elements

$$D_{jk}(\mathbf{R}) = \exp\left[\frac{\pi i}{N}Ln_1(n_2 + 2j)\right] \delta_{j,k-n_2}; \quad (\text{mod } N); \quad j, k = 0, 1, \dots, N-1. \quad (7)$$

The factor system for this representation agrees with that given by (5) since

$$D(\mathbf{R})D(\mathbf{R}') = D(\mathbf{R} + \mathbf{R}') \exp\left[\frac{\pi i}{N}L(n_2n'_1 - n_1n'_2)\right].$$

Note that all factors are roots of 1 of the order  $2N$ , whereas the dimension of the considered representations is  $N$ . Therefore, there exists an equivalent normalized (and standard) factor system  $m$ , *i.e.* such a system that all factors are the  $N$ -th roots of 1 [10]. It can be obtained if each matrix  $D(\mathbf{R})$  will be multiplied by

$\phi(\mathbf{R}) = \exp(-\pi i L n_1 n_2 / N)$ . In this way new (and nonequivalent to the previous ones) irreducible representations are obtained

$$\langle N, L; \mathbf{0} \rangle_{jk} [n_1, n_2] = \delta_{j, k-n_2} \omega_N^{L n_1 j}, \quad j, k = 0, 1, \dots, N-1, \quad \omega_N = \exp(2\pi i / N), \quad (8)$$

where  $\langle N, L; \mathbf{0} \rangle$  denotes the representation (the role of the zero vector  $\mathbf{0}$  will be explain below) and a vector  $n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$  was replaced by a pair  $[n_1, n_2]$ . Since only the unique element  $[0, 0]$  has nonzero character, then this representation is irreducible (as projective representation of the translation group  $T_N \simeq \mathbb{Z}_N \otimes \mathbb{Z}_N$ ). It is easily shown that a factor system

$$m_N^{(L)}([n_1, n_2], [n'_1, n'_2]) = \omega_N^{L n_2 n'_1} \quad (9)$$

corresponds to this form of representations. It can be shown (see [11]; more detailed discussion of different gauges is in preparation) that this factor system corresponds to the Landau gauge, used in many papers (see, *e.g.*, [7, 12]). Three important facts should be stressed:

- Projective representations with different, though equivalent, factor systems are *nonequivalent* [10], so representations discussed in this paper and those introduced by Brown are nonequivalent. However, the same set of basis function can be used.
- In fact, modification of the factor system (5) corresponds to different choice of the gauge  $\mathbf{A}$ ; it was shown [11] how to introduce the magnetic translations for any gauge  $\mathbf{A}$  (such that  $\mathbf{H} = \mathbf{rot} \mathbf{A}$ ).
- Equivalent factor systems lead to the same expression for the commutator

$$D(\mathbf{R})D(\mathbf{R}')D^{-1}(\mathbf{R})D^{-1}(\mathbf{R}') = \omega_N^{-L(n_1 n'_2 - n_2 n'_1)}.$$

The actual form of basis function is not discussed here (see, *e.g.*, [2, 3, 5, 13] for more details). It is worth noting that these functions, denoted as  $|s\rangle$  with  $s = 0, 1, \dots, N-1$ , are eigenfunctions of  $\langle N, L; \mathbf{0} \rangle [n_1, 0]$  operators, whereas the operators  $\langle N, L; \mathbf{0} \rangle [0, n_2]$  permutes them in a cyclic way (*cf.* [4, 14]):

$$\langle N, L; \mathbf{0} \rangle [n_1, 0] |s\rangle = \omega_N^{L n_1 s} |s\rangle; \quad (10)$$

$$\langle N, L; \mathbf{0} \rangle [0, n_2] |s\rangle = |s - n_2\rangle \pmod{N}. \quad (11)$$

The special choice of  $\langle N, L; \mathbf{0} \rangle$  put  $\mathbf{a}_1$  and  $\mathbf{a}_2$  on a different footing.

Let us assume that the number  $L$  has a common factor with  $N$ , say  $L = l\nu$  and  $N = n\nu$  with  $\nu = \gcd(L, N) > 1$  (please recall that  $L$  is simply connected with the magnetic field magnitude). It is easy to notice that the *magnetic* periodicity is obtained for the smaller period  $n$  and the factor system (9) can be written as

$$m_n^{(l)}([n_1, n_2], [n'_1, n'_2]) = \omega_n^{l n_2 n'_1}. \quad (12)$$

Therefore, one may consider factor systems (12) for all divisors  $n$  of  $N$  and  $l$  mutually prime with  $n$  (*i.e.*  $\gcd(n, N) = n$  and  $\gcd(l, n) = 1$ ). It is an easy task of combinatorics to show that  $N$  different factor systems, corresponding to  $L = 0, 1, \dots, N-1$ , are obtained in this way [15]. Since even for  $n < N$  the  $N \times N$  lattice is still under the question, so  $N$  will be called hereafter the *crystal* period, whereas  $n$ , for which  $T(n\mathbf{R}) = \mathbf{1}$ , will be called the *magnetic* period. Hence, the  $N \times N$  lattice can be viewed as a  $\nu \times \nu$  lattice, with the translation group  $T_\nu = \mathbb{Z}_\nu \otimes \mathbb{Z}_\nu$ , of  $n \times n$  magnetic cells. Let  $\mathbf{q} = [q_1, q_2]$  and

$$\langle \mathbf{q} \rangle_\nu [\xi_1, \xi_2] = \exp[-2\pi i (q_1 \xi_1 + q_2 \xi_2) / \nu] = \omega_\nu^{-(q_1 \xi_1 + q_2 \xi_2)} \quad (13)$$

be the irreducible representation of  $T_\nu$ . Then it is easy to check that  $n$ -dimensional matrices

$$\langle n, l; \mathbf{q} \rangle_{jk} [n_1, n_2] = \delta_{j, k - \eta_2} \omega_n^{l\eta_1 j} \omega_\nu^{- (q_1 \xi_1 + q_2 \xi_2)} \quad (14)$$

form a projective irreducible representation of the group  $T_N$  with a factor system (12). In this formula  $[\xi_1, \xi_2]$  labels magnetic cells, whereas  $[\eta_1, \eta_2]$  labels positions within a magnetic cell, *i.e.*  $n_i = \eta_i + \xi_i n$ . As the basis function the eigenvectors  $|n, l; \mathbf{q}; s\rangle$ ,  $0 \leq s < n$ , of the matrix  $\langle n, l; \mathbf{q} \rangle [1, 0]$  will be used. The character of the representation (14) is easily calculated as

$$\chi \langle n, l; \mathbf{q} \rangle [n_1, n_2] = \delta_{\eta_1, 0} \delta_{\eta_2, 0} n \omega_\nu^{- (q_1 \xi_1 + q_2 \xi_2)}. \quad (15)$$

For given  $n$  and  $l$  (*i.e.* for a given factor system) we obtained  $\nu^2$  nonequivalent irreducible projective representations (labeled by  $\mathbf{q}$ ), so we obtained all of them [10]. In particular we have

$$\langle 1, 1; \mathbf{q} \rangle = \langle \mathbf{q} \rangle_N, .$$

### 3 Products of irreducible projective representations

Let us consider a product of two projective representations  $T$  and  $T'$  of a given group  $G$  with factors systems  $m$  and  $m'$ . For a matrix element of the considered product we have  $(T \otimes T')_{ij, kl}(g) = T_{ik}(g) T'_{jl}(g)$  so

$$\begin{aligned} [(T \otimes T')(g)(T \otimes T')(g')]_{ij, kl} &= \sum_{p, q} (T \otimes T')_{ij, pq}(g) (T \otimes T')_{pq, kl}(g') \\ &= \sum_{p, q} T_{ip}(g) T'_{jq}(g) T_{pk}(g') T'_{ql}(g') \\ &= m(g, g') m'(g, g') T_{ik}(gg') T'_{jl}(gg') \\ &= m''(g, g') (T \otimes T')_{ij, kl}(gg'), \end{aligned} \quad (16)$$

where  $m''(g, g') = m(g, g') m'(g, g')$  is, in general, a new factor system. (Of course a product of two vector representations is a vector representation.) In the considered case all factors are the  $N$ -th root of 1 and a product of two factors  $m_n^{(l)}$  and  $m_{n'}^{(l')}$ , is equal to

$$m([n_1, n_2], [n'_1, n'_2]) = \omega_N^{(l\nu + l'\nu') n_2 n'_1}, \quad (17)$$

so it corresponds to representation with  $L = l\nu + l'\nu'$ . It means that the set of factor systems (12) is closed with respect to the multiplication and, therefore, the representations (14) and their direct sums form a closed set with respect to the tensor product. Of course, we can add representations with the same factor system only and vice versa — if a given projective representation with factor system  $m$  is reducible then it can be decomposed into a direct sum of irreducible projective representations with the same factor system  $m$ . Moreover, the orthogonality relations for representations and their characters are valid for representations with the same factor system [10]. Hence, one must be very careful decomposing a given projective representation — representations with different factor systems *can not* be compared.

Let  $D$  be a product of two irreducible representations  $\langle n, l; \mathbf{q} \rangle$  and  $\langle n', l'; \mathbf{q}' \rangle$ . Its factor system is given by (17), its dimension equals  $nn'$  and its character is

$$\chi^D [n_1, n_2] = \delta_{\eta_1, 0} \delta_{\eta_2, 0} \delta_{\eta'_1, 0} \delta_{\eta'_2, 0} n n' \omega_N^{-n(q_1 \xi_1 + q_2 \xi_2) - n'(q'_1 \xi'_1 + q'_2 \xi'_2)}, \quad (18)$$

so it is nonzero only for  $n_i = x_i m$ , where  $m = nn'/\gamma$ ,  $\gamma = \gcd(n, n')$ ,  $0 \leq x_i < \mu = N/m = \gcd(\nu, \nu')$ . Substituting  $m$  and  $\mu$  to the above formula one easily obtains

$$\chi^D[n_1, n_2] = \delta_{\eta_1, 0} \delta_{\eta_2, 0} m \gamma \omega_\mu^{-(q_1 + q'_1)x_1 - (q_2 + q'_2)x_2}; \quad (\text{mod } m). \quad (19)$$

Since  $\nu/\mu = n'/\gamma$  then  $L$  in (17) can be written as

$$L = \mu \left( \frac{l\nu}{\mu} + \frac{l'\nu'}{\mu} \right) = \mu \left( \frac{ln'}{\gamma} + \frac{l'n}{\gamma} \right) = \mu \Lambda. \quad (20)$$

It seems that this determines a factor system  $m_m^{(\Lambda)}$ . However, it is impossible to exclude a priori the case when  $\gcd(\Lambda, m) = \ell > 1$ . It is evident that the summands in (20) have no common factor, but it may happen that their sum  $\Lambda$  has a common factor with  $m$ . Therefore, the considered product has to be decomposed into irreducible representations with a factor system  $m_M^{(\lambda)}$ , where  $\lambda = \Lambda/\ell$  and  $M = m/\ell$ . The scalar product of appropriate characters gives us

$$\begin{aligned} f(\langle M, \lambda; \mathbf{k} \rangle, \langle n, l; \mathbf{q} \rangle \otimes \langle n', l'; \mathbf{q}' \rangle) &= \frac{mM\gamma}{N^2} \sum_{x_1, x_2=0}^{\mu-1} \omega_\mu^{-(q_1 + q'_1 - k_1)x_1 - (q_2 + q'_2 - k_2)x_2} \\ &= \frac{\gamma}{\ell} \delta_{k_1, q_1 + q'_1} \delta_{k_2, q_2 + q'_2}; \end{aligned} \quad (21)$$

there are  $\ell^2$  such representations with  $k_i = q_i + q'_i \text{ mod } \mu$ .

The most interesting is the case when  $n = n'$  and  $l = l'$ , since  $n$  and  $l$  are determined by the magnetic field magnitude (and the crystal period  $N$ ). Therefore, representations  $\langle n, l; \mathbf{q} \rangle$  and  $\langle n, l; \mathbf{q}' \rangle$  act in two  $n$ -dimensional eigenspaces of one-electron states and their product should correspond to two-electron space of states. In this case one obtains that the resultant representation is  $n^2$ -dimensional and  $\gamma = m = n$ ,  $\mu = \nu$ . From (19) the character is equal to

$$\chi^D[n_1, n_2] = \delta_{n_1, x_1 n} \delta_{n_2, x_2 n} n^2 \omega_\nu^{-(q_1 + q'_1)x_1 - (q_2 + q'_2)x_2} \quad (22)$$

with  $0 \leq q_i, q'_i, x_i < \nu$ . The factor system is given by (17):

$$m([n_1, n_2], [n'_1, n'_2]) = \omega_n^{2ln_2n'_1} = \omega_n^{\Lambda n_2 n'_1}, \quad (23)$$

where, see (20),  $\Lambda = 2l$ . At this moment the cases of odd and even  $n$  have to be considered separately. In the first case  $\ell = \gcd(n, 2l) = 1$  and the obtained representations decomposes into  $n$  copies of the representation  $\langle n, 2l; \mathbf{k} \rangle$  with  $k_i = (q_i + q'_i) \text{ mod } \nu$ . In the second case, however,  $\ell = 2$  and  $M = n/2$  so the considered product decomposes into representations  $\langle n/2, l; \mathbf{k} \rangle$ : there are four representations with  $k_i - q_i - q'_i = 0, \nu$  and each of them appears  $n/2$  times. In a similar way, the coupling of  $d$  representations  $\langle n, 1; \mathbf{q}^{(j)} \rangle$ ,  $j = 1, 2, \dots, d$  with  $n = dM$  changes the magnetic period from  $n$  to  $M$ , however it has not been caused by modification of the magnetic field but by multiplication of the charge by  $d$  [see (6)].

Clebsch-Gordan coefficients for the considered product are strongly ambiguous since the frequencies of irreducible representations can be very large so they are not discussed here.

## 4 Example

Let us consider  $N = 12$  and two representations:  $\langle 3, 1; [1, 0] \rangle$  and  $\langle 6, 1; [1, 0] \rangle$ . The corresponding co-divisors of  $n = 3$  and  $n' = 6$  are  $\nu = 4$  and  $\nu' = 2$ , respectively, so the greatest common divisor  $\gamma = 3$  and the least common multiplicity is  $m = 6$ .

Hence, a co-divisor  $\mu = N/6 = 2$ . Now we have to calculate  $L$  from Eq. (17) and to write as a multiplicity of  $\mu$ , see (20). It is easy to obtain that

$$L = 4 + 2 = 6 = 2 \cdot 3 \Rightarrow \Lambda = 3.$$

So,  $\ell = \gcd(\Lambda, m) = 3$  and the considered product decomposes into nine irreps. To determine them one has to find

$$M = \frac{m}{\ell} = \frac{6}{3} = 2 \quad \text{and} \quad \lambda = \frac{\Lambda}{\ell} = \frac{3}{3} = 1.$$

Therefore, a factor system of obtained irreducible representations should be denoted as  $m_2^{(1)}$  instead of  $m_6^{(3)}$  and these representations are two-dimensional. Due to the condition  $k_i = q_i + q'_i \bmod \mu$  one obtains that  $k_i$ ,  $i = 1, 2$ , should be even. Since  $N/M = 6$  then  $k_i = 0, 1, 2, 3, 4, 5$  and, eventually, the following decomposition can be written

$$\langle 3, 1; [1, 0] \rangle \otimes \langle 6, 1; [1, 0] \rangle = \bigoplus_{k_1=0,2,4} \bigoplus_{k_2=0,2,4} \langle 2, 1; [k_1, k_2] \rangle.$$

#### 4.1 Remarks

To obtain the *magnetic periodicity* for  $n = 3$ ,  $l = 1$  and  $n' = 6$ ,  $l' = 1$  one has to assume [see Eq. (6)] that

$$h = \frac{2\pi}{\Omega} \frac{l}{n} a_3 = \frac{1}{3} \frac{2\pi}{\Omega} a_3 \quad (24)$$

and

$$h' = \frac{2\pi}{\Omega} \frac{l'}{n'} a_3 = \frac{1}{6} \frac{2\pi}{\Omega} a_3. \quad (25)$$

It seems that there are two different magnitudes of the magnetic field since  $h \neq h'$ . But putting

$$h = q \frac{e}{\hbar c} H \quad \text{and} \quad h' = q' \frac{e}{\hbar c} H \quad (26)$$

with

$$q = 2q' \quad (27)$$

the same value of  $H$  is obtained

$$H = h' \frac{\hbar c}{e} \frac{1}{q'} = \frac{1}{6} \frac{2\pi}{\Omega} a_3 \frac{\hbar c}{e} \frac{1}{q'} = \frac{1}{3} \frac{2\pi}{\Omega} a_3 \frac{\hbar c}{e} \frac{1}{2q'} = h \frac{\hbar c}{e} \frac{1}{q} = H$$

The resultant representations correspond to  $M = 2$ ,  $\lambda = 1$  so

$$H = \underbrace{\frac{2\pi}{\Omega} \frac{\lambda}{M} a_3}_{h''} \frac{\hbar c}{e} \frac{1}{q''} = \frac{1}{2} \frac{2\pi}{\Omega} a_3 \frac{\hbar c}{e} \frac{1}{q''}.$$

Since in all three cases  $H$  is the same and the charges  $q, q', q''$  have to be integers then

$$\frac{1}{2q''} = \frac{1}{6q'} = \frac{1}{3q}.$$

Substituting  $q = 2q'$  one obtains  $2q'' = 6q'$  and

$$q'' = 3q' = q + q',$$

so multiplication of representations corresponds to addition of charges.

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